Analogy between scale symmetry and relativistic mechanics. II. Electric analog of turbulence

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In turbulent structure functions, we separate a nonuniversal part, depending on the most frequent events, and a universal part, i.e., reduced structure functions. We focus on the universal contribution using only scale symmetry arguments and an analogy with electricity developed in our companion paper. A conservation law leads to a general scaling and a method of computation of the reduced structure functions. Around the infinite-Reynolds-number limit, we propose a perturbation development which is both regular and compatible with scale symmetry. This linear approximation accounts for a large variety of turbulent flows, from jet to boundary layer turbulence. [S1063-651X(97)12711-8]

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I. INTRODUCTION

In homogeneous isotropic turbulence, the velocity field $\vec{u}(\vec{x})$ is random, and a central question is the following: In its statistics, what is universal, and what depends on a specific experiment?

In the quest for universal quantities, one often extracts the longitudinal velocity increments over a distance ℓ,

$$\delta u_{\ell}=[\vec{u}(\vec{x}+\vec{\ell})-\vec{u}(\vec{x})].\frac{\vec{\ell}}{\ell}, \tag{1}$$

and considers its successive moments [1]. They vary with the scale ℓ, and one defines the scaling exponent of the nth moment as $d\ \ln(\langle|\delta u_{\ell}|^{n}\rangle)/d\ \ln\ell$. If δu_{ℓ} was a power law of ℓ, this exponent would be a constant; here, we are interested in the variation of this local exponent with the scale.

In Refs. [2,4,5], we tried to determine a possible universal behavior for the scaling exponents of the moments of the velocity increments, using symmetry arguments. Inspired by the pioneering work of Nottale [6], in Refs. [4,2] we developed an exponent relativity formalism to derive a relation for scaling exponents of successive moments, in the case of a scale-invariant homogeneous random field. As shown in Ref. [5], this formalism enables a classification of the possible statistics of a scale-invariant process, in terms of topological quantities, namely, the value of the minimal and maximal multifractal exponents.

Can we also use scale symmetry to constrain the variation of the structure functions with scales? That is, having already focused on the link between successive moments, can we now turn to the link between different scales? Dubrulle [7] began it, using a linear amplitude equation. However, symmetry allows more general, nonlinear equations, via Lagrangian dynamics. Castaing [8] thoroughly discussed what would happen if turbulence was governed by a Lagrangian formalism, whatever the precise choice of any Lagrangian: in particular, there would exist a conserved quantity $\mathcal{P}$, associated to scale invariance through Noether's theorem. At finite Reynolds number such a scale-independent ''trajectory invariant'' could play the same fundamental role as the one Kolmogorov assigned to energy dissipation, using intuitive dimensional considerations [9].

Motivated by this vision, we tried, in our companion paper [10], to derive a possible Lagrangian formalism for scale-invariant random systems, in analogy with relativity and electricity. In the present paper, our aim is now to focus on fully developed hydrodynamical turbulence, still using only symmetry arguments. This means we leave open the determination of the exact Lagrangian, which depends *a priori* on the detailed structure of the given flow: geometry, forcing, and Reynolds number. There are now growing experimental indications that turbulent velocity increments follow, if not exactly a log-normal statistics [8,3], at least a statistics very close to log-normal: for example, fits of the scaling exponents ζ_n with a log-Poisson law $\zeta_n \sim C(1-\beta^n)$ [11–13], in various recent experimental configurations, are obtained with a β parameter $\beta=0.95\pm0.06$ [14], whereas the log-normal statistics requires $\beta=1$. For the sake of simplification, in the present paper we only consider the case where turbulence is log-normal. Other types of statistics could also be considered, if evidence of subsequent deviations from log-normality could be experimentally achieved, e.g., in nonhomogeneous and/or nonisotropic turbulence, or other types of systems (see the Appendix).

II. DETERMINATION OF STRUCTURE FUNCTIONS

A. Notations and analogy

For simplicity, we focus on the case of a positive field [4]. We thus consider only absolute values of the velocity increments $|\delta u_{\ell}|$. For reasons which will appear later in the particularly clear shape of Eqs. (4) and (5), we now consider the quantity δu_{ℓ}^{0} defined as an average over the logarithm, as follows:

$$\ln(\delta u_{\ell}^{0})\equiv\langle\ln|\delta u_{\ell}|\rangle. \tag{2}$$

Physically, δu_ℓ^0 characterizes the most probable velocity increment. If $\delta u_\ell^0 \sim \ell^{\Delta_0}$, it provides a linear contribution $n\Delta_0$ to the scaling exponents of the nth moment of the velocity increments. Of course, if the turbulence is purely log-normal, $\Delta_0 = 0$. Most intermittency models, however, use $\Delta_0 \neq 0$. For example, the Kolmogorov 1941 and 1962 theories correspond to $\Delta_0 = \frac{1}{3}$. This value can be traced back to the famous Kolmogorov refined similarity hypothesis, linking statistics of velocity increments and energy transfers, and which is a phenomenological generalization of the Kolmogorov $\frac{4}{5}$ law (20). Clearly, δu_ℓ^0 is a nonuniversal quantity, which depends on the particular size and set-up of a given experiment. In Sec. IV B we show how the Kolmogorov $\frac{4}{5}$ law can be used to obtain the scale dependence of δu_ℓ^0, once the log-normal part has been determined.

For the time being, we shall set aside the problem of finding the exact shape of δu_ℓ^0, and concentrate on the log-normal part of the processes by considering reduced structure functions of order n:

$$S_n(\ell) = \frac{\langle |\delta u_\ell|^n \rangle}{(\delta u_\ell^0)^n}. \tag{3}$$

For that purpose, we introduce the log coordinates

$$X_n(T) = \frac{d \ln S_n(\ell)}{dn},$$

$$T = \ln\left(\frac{\ell}{L}\right). \tag{4}$$

Here L is an arbitrary scale of reference; e.g., without loss of generality, we can take L as the Kolmogorov scale η so that scales with $T > 0$ are representative of the inertial range, while scales with $T < 0$ trace the dissipative scales. These notations (X_n, T) are not innocent, and the scaling exponent of a process relative to another one is noted in analogy with a velocity:

$$\dot{X}_n = \frac{dX_n}{dT}. \tag{5}$$

In our companion paper [10], we discussed at length the constraints set by scale symmetry on scaling exponents; however, if the reduced velocity increments $|\delta u_\ell| / \delta u_\ell^0$ have a log-normal statistics, the formalism applying to log coordinates are particularly simple [16]:

$$X' = X - VT,$$

$$T' = T, \tag{6}$$

$$\dot{X}' = \dot{X} - V.$$

The nth log coordinate $X' = X_n$ can then be obtained as the transform of the first log coordinate $X = X_1$ in an accelerated reference frame moving at the velocity $V = \dot{X}_{n-1}$.

B. Electric field and breaking of symmetries

In the case of scale-invariant processes, we explained why the Lagrangian dynamics should depend on only one quantity E [see Eq. (7)] below. It is a (purely formal) analog of an electric field driving a charged particle with a coupling constant e/m. Symmetry arguments are thus a source of inspiration for new model even, as in the log-normal case, when constraints are less severe [17]. In a system forced by an arbitrary external E, two basic symmetries can be broken: the T-translation symmetry, and/or the X-translation symmetry. In turbulence, as in any physical experiment, finite size imposes upper and lower cutoffs, i.e., extreme limits on the scale which break the T-translation symmetry, i.e., the scale symmetry. By contrast, the X-translation symmetry, i.e., the invariance of the physical laws with respect to changes of amplitude of the *relative* velocity fluctuations, is associated with the concept of a ''scale-invariant cascade of energy.'' This is in the spirit of Kolmogorov [9] or Castaing [8], and implies that there is a conserved quantity along the scale, the generalized impulsion $\mathcal{P}$, derived in Ref. [10] and associated with this X-translation symmetry preserved in turbulence. We show below that this conservation gives rise to a property named general scaling [15].

C. Dynamics

We wish to know how the reduced structure function of order n varies with the scale. In our analogy, this amounts to determining the log coordinate X_n. This is governed by the equations

$$\frac{d\dot{X}_1}{dT} = \alpha E(T),$$

$$\frac{d\dot{X}_n}{dT} = \alpha E(T) + \frac{d\dot{X}_{n-1}}{dT}. \tag{7}$$

Here, E is the analog of an ''electric field,'' and describes the deviations with respect to exact scale symmetry (see Sec. III). α is a coupling constant, describing the reaction of the turbulent flow to the scale symmetry-breaking terms E. The first equation is the Euler-Lagrange equation for X_1 written in a fixed reference frame. The second is the same equation written in an accelerated frame of reference (see Sec. II A), in which the second term on the right-hand side accounts for the inertial force. By an immediate recurrence, one therefore obtains the equation of evolution for any n:

$$\frac{d\dot{X}_n}{dT} = n\alpha E(T). \tag{8}$$

The quantity X_n is known once three quantities are given.

(i) Its coupling constant $n\alpha$.

(ii) Its initial values of X_n and $\dot{X}_n$ at, say, $T=0$. The initial value of X_n is noted $X_n(0)$. It contributes via a prefactor $\int_0^n X_p(0)dp$ to the structure function, and is usually fixed by the large-scale injection mechanism. Similarily, we note that $\dot{X}_n(0) = n\dot{X}_1(0)$, the initial value of $\dot{X}_n$, is also fixed by the injection mechanism, i.e., the experimentalist.

(iii) The electric field E itself. This cannot be derived from mere symmetry considerations, and should be either determined from experiments, or computed using another systematic theory. However, as we now show, an interesting conclusion can be reached regardless of the specific shape of E.

D. General scaling

Let us consider relation (8). Using the initial conditions and the X independence of E, it can be integrated into:

$$X_n - X_n(0) = n[X_1 - X_1(0)]. \tag{9}$$

Integrating Eq. (9) with respect to n yields a relation between reduced structure functions of order n and 1:

$$\ln\left(\frac{S_n(\ell)}{S_n(\eta)}\right) = n^2 \ln\left(\frac{S_1(\ell)}{S_1(\eta)}\right). \tag{10}$$

Such a result has an interesting consequence: the usual structure functions $\langle|\delta u_\ell|^n\rangle$ obey a general scaling extending throughout the whole range of scale under the form

$$\ln\left(\frac{\langle|\delta u_\ell|^n\rangle}{\langle|\delta u_\ell|^3\rangle^{n/3}}\right) = \frac{n^2-3n}{p^2-3p}\ln\left(\frac{\langle|\delta u_\ell|^p\rangle}{\langle|\delta u_\ell|^3\rangle^{p/3}}\right). \tag{11}$$

Property (11) was already experimentally observed in turbulence by Benzi *et al.* [15]. It was named ''general scaling'' because it is valid from the injection scale down to the smallest scale resolved into the system, and thus generalizes the self-similar properties of the structure functions in the inertial range. Here we have shown that it stems directly from conservation along the scales of the generalized impulsion, in the log-normal case. We do not actually know if this property still holds for statistics other than log-normal. Checking this requires a nontrivial ''general relativity'' type of computation; see the discussion in our companion paper. Even if this were not rigorously true, one expects to see Eq. (11) hold only approximatively with a slightly different constant, if the turbulence is nearly log-normal. In any case, the general scaling property provides a significant simplification of the determination of the reduced structure functions, because only the shape of $S_1(\ell)$ is needed. Below we show how to compute this in a simple case.

III. DETERMINATION OF $S_1(\ell)$

A. Basic equations

The general scaling property (11) means that variations of the reduced structure functions are completely determined by the variations of X_1 in a fixed reference frame. These variations can be found by solving the equations

$$\frac{d\dot{X}_1(T)}{dT} = \alpha E(T). \tag{12}$$

The electric field itself is linked to the current J, via the Maxwell equation [10]

$$\frac{\partial E}{\partial T} = -J. \tag{13}$$

In a given experiment, physical conditions select the shape of J, and then the shape of the structure functions via Eq. (12). The major challenge is now to relate J and E to physical quantities characterizing an experiment, i.e., to find a closure relation complementing Maxwell's Eq. (13).

In the case where X translation symmetry holds, E and J are functions of T only. Thus the theorem of implicit functions implies that one can write J as a function of E, i.e., $J = J(E)$. In a perfectly scale invariant system, i.e., in infinite Reynolds number turbulence), $\dot{X}_1$ is constant, and E is zero. In the vicinity of the inertial range, or when the Reynolds number Re is large enough, the system is sufficiently close to the scale-invariant situation, and E is small. Thus one can expand J as a function of the parameter E, a standard procedure in electricity [18]. This expansion is a perturbative development around the Re$=\infty$ limit.

Usually, in this Re$=\infty$ limit, the quantity which is developed is the structure function, or the scaling exponents: in any case it is a development of the solutions. But boundary conditions on T (i.e., upper and lower cutoffs on the scale ℓ) are rejected at infinity in the idealized Re$=\infty$ limit, while they are necessarily finite in a real, experiment at finite Re. This implies that the classical method suffers from two drawbacks: (i) since boundary conditions are qualitatively different, the development in 1/Re is generally singular at 1/Re $=0$; and (ii) the solution found at finite Re has no reason to be the solution of a scale-symmetric equation.

Alternatively, we develop the equation itself instead of the solution. In that way we are insensitive to the precise location of any precise scale which could appear, whether cutoffs or crossovers. We do not have to assume *a priori* the existence of an inertial range, of a Kolmogorov length, or of a large ratio between largest and smallest accessible scales. By construction, our development is (i) regular at 1/Re$=0$, and (ii) compatible with symmetry requirements. Moreover, we do not have yet to specify the precise functional dependence of our equations in 1/Re; this will be discussed below.

B. Ohmic case

When E is identically zero, J is also zero, as a consequence of Eq. (13). Thus we now consider the leading term

$$J = \sigma E. \tag{14}$$

This is the linear, or ''Ohmic,'' case, and σ is the analog of a conductivity. In electricity, different material possess different (or even nonlinear) conductivities: similarly, we can expect different turbulent flow to be characterized by different ''conductivities,'' and to belong to different classes of solutions, possibly not even linear. Of course, nothing forbids nonlinear corrections to Eq. (14). But we will see that this dominant linear physics is rich enough, and is probably representative of a wide variety of high-Re turbulent flows.

The system of equations (12)–(14) is closed. We have to introduce two integration constants $E_0 = E(0)$ and $\dot{X}_1(0)$ which will lead the discussion. Resolution is simple:

$$E(T) = E_0 e^{-\sigma T}, \tag{15}$$

TABLE I. Fits of various turbulent configurations following the procedure described in the text. The three parameters are, respectively, σ, μ, and $\alpha E_0/\sigma$, except in the case of stretched exponential where they are $\sigma=0$, $\dot{X}_1(0)$, and $\alpha E_0/2$. R_λ is the Reynolds number based on the Taylor microscale, R is the pseudo-Reynolds number defined as the ratio of the integral scale to the Kolmogorov scale. References of data: wind tunnel [22], turbulence in helium [23], jet turbulence [24], turbulence behind a cylinder [25], and numerical simulation [26].

Experiment	R_λ	R	Solution	First parameter	Second parameter	Third parameter
Wind tunnel	2500	40000		0.82 ± 0.04	0.76 ± 0.02	-17.8 ± 0.9
Helium	2313	12157		0.78 ± 0.06	0.86 ± 0.02	-15.4 ± 1.7
Jet	800	1250	stretched exp.	0.18 ± 0.01	0	-9.93 ± 0.13
Behind cylinder	470	714	degenerate	0	1.37 ± 0.14	-0.18 ± 0.04
Simulation	335	85		0.56 ± 0.04	0.43 ± 0.05	-9.75 ± 0.1

$$\dot{X}_1(T)=\dot{X}_1(0)+\frac{\alpha E_0}{\sigma}(1-e^{-\sigma T}),$$

$$X_1(T)-X_1(0)=\dot{X}_1(0)T+\frac{\alpha E_0}{\sigma^2}(e^{-\sigma T}-1+\sigma T).$$

Let us now discuss this solution of the Ohmic equations; remember that it is expressed in log variables. See Eq. (4) for their definition.

C. Ohmic solution

First of all, we want to link this solution with the usual experimental observation of an ''inertial range,'' i.e., a range of scales where structure functions are approximately power laws of the scale. In our log variables, this is an interval of T where X is linear in T. Does it exist at all?

The answer is of course yes. On the right-hand side of Eq. (15c), terms linear in T dominate higher-order terms if $|e^{-\sigma T}|\ll 1$ or $|\sigma T|\ll 1$. Thus there are two points in our discussion, and two special cases are examined.

(i) As long as $|T|\ll|\sigma^{-1}|$, Eq. (15c) writes $X_1(T)-X_1(0)\sim\dot{X}_1(0)T$ up to T^2 terms. When σ tends to zero, this range widens at the expense of the next one. Exactly when $\sigma=0$, $E=E_0$ is constant, and the solution is simply

$$X_1(T)-X_1(0)=\dot{X}_1(0)T+\frac{\alpha E_0}{2}T^2. \tag{16}$$

Such a solution was also found using a normal form approach by Dubrulle [7]. Only when E is strictly zero does this solution yield an exact power law.

(ii) There is a well distinct linear range, with $|e^{-\sigma T}|\ll 1$, at large scales, i.e., $T>\sigma^{-1}$ if σ is positive [19]. Upwards, the inertial range extends up to the ''initial condition:'' the upper cutoff, usually the scale at which the experimentalist injects energy. We see in Eq. (15b) that the corresponding exponent is $\mu=\dot{X}_1(0)+\alpha E_0/\sigma$ for the reduced structure function of order 1, and $n^2\mu$ for order n.

When σ increases, this range widens at the expense of the preceding one. In the limit $\sigma\to\infty$, Eqs. (15) reduce to $E\equiv 0$ and $\dot{X}_1=$const, which is an exact power law.

D. Physical interpretation

Clearly, the idealized case of an infinite Reynolds number, where structure functions are exact power laws of the scale, must be identified with $E\equiv 0$. However, that does not set any constraints on σ. Fits to experimental data (see Table I) indicate low values of σ. This is compatible with physical [8] and symmetry [7] arguments, which suggest that the asymptotic behavior toward this idealized limit is of the shape $\sigma\sim 1/\ln(\mathrm{Re})$. This would mean that we must identify the inertial range with the range $T>\sigma^{-1}$, and take the triple limit where σ^{-1}, E_0, and the size of accessible scale range tends together to zero. If this were true, the dissipative range would have a width of order $2\sigma^{-1}$.

In the particular case where the exponent $\mu=0$, we obtain an especially interesting solution:

$$X_1(T)-X_1(0)=\frac{\alpha E_0}{\sigma^2}(e^{-\sigma T}-1). \tag{17}$$

It is a ''stretched exponential'' when expressed in real variables ℓ,S:

$$S_n(\ell)=S_n(\eta)\ \exp\left[n^2\alpha E_0\frac{(\ell/\eta)^{-\sigma}-1}{\sigma^2}\right]. \tag{18}$$

This, then, is exactly the solution proposed in Ref. [20] on the basis of experimental results. This solution was also derived using a Lagrangian approach based on scale-invariance symmetry by Castaing [8], and using a normal form approach by Dubrulle [7]. This solution is a true exponential if $\sigma=1$, and becomes a power law in the limit σ^{-1}, $E_0\to 0$. It is a generic solution of the scale-symmetric equation, and thus has a similar mathematical role, when the accessible scale range is finite, as the celebrated power law plays when cutoffs are rejected at infinity.

E. Local scale covariance

This interpretation becomes more exciting in the light of local symmetry requirements. As first discussed by Pocheau [21], local scale symmetry requires an invariance of the structure function by arbitrary changes of scale resolution. Dubrulle [7] pointed out that this amounts to symmetry under the transformation

$$T \to aT, \quad \ln(R) \to a \ln(R),$$

$$\ln \langle (\delta u_\ell)^n \rangle \to a \ln \langle (\delta u_\ell)^n \rangle, \tag{19}$$

where a is a positive real number and $\ln R = \ln(\ell_0/\eta)$ is a pseudo-Reynolds number, defined via the ratio of the injection scale ℓ_0 to the Kolmogorov scale η. This symmetry imposes that $X_1/\ln(R)$ is a function of $T/\ln(R)$ only. In the Ohmic case, this is only possible provided $\sigma \propto 1/\ln(R)$, $\alpha E_0 \propto 1/\ln(R)$, and $\dot{X}_1(0)$ is independent of R.

When this is satisfied, the stretched exponential [Eqs. (17) and (18)] tends smoothly toward the perfect self-similar $R=\infty$ solution, as is often discussed in literature devoted to turbulence theory. However, surprisingly, the scaling exponents *do not* tend toward their Kolmogorov expression $\zeta_n = n/3$. Since $\dot{X}_1(0) = -\alpha E_0/\sigma$ keeps its finite value, the log-normal character subsists in the limit of infinite Reynolds number.

In the more general case (15), however, it remains possible to tend continuously toward the Kolmogorov solution if and only if $\dot{X}_1(0)=0$ and σ and αE_0 tend together to zero to the same order in Reynolds number.

IV. ANALYSIS OF TURBULENT DATA

A. Fitting experimental data

To investigate the domain of validity of the ohmic approximation, we computed the function $X_1(T)$ in various turbulent configurations, at various Reynolds numbers. The following procedure was used.

(i) Check that the general scaling (11) holds.

(ii) In case it holds, compute the flatness function $F = \ln(\langle \delta u_\ell^4 \rangle / \langle \delta u_\ell^2 \rangle^2)$. Because of the general scaling property, this function is just eight times the function $X_1(T)$.

(iii) Define a ''log scale T_i typical of the inertial range'' as the location where $d \ln \langle \delta u_\ell^3 \rangle / d \ln \ell = 1$.

(iv) Find the local scaling exponent $\mu = \frac{1}{8}\, dF/dT$ at $T = T_i$.

(v) Compute $F/8\mu$, a normalized version of X_1 which facilitates comparisons between different configurations.

(vi) Fit the obtained result by the Ohmic solution (15) using three parameters: $\sigma, \mu = \dot{X}_1(0) + \alpha E_0/\sigma$, and $\alpha E_0/\sigma^2$.

(vi) If the fitted value of μ or σ is much smaller than 1, another fit is performed setting this parameter to zero, and refitting the two other parameters via the stretched exponential (17) or the degenerate (16) solution.

(vii) Keep the solution which yields a better fit to the data (according to a χ^2 test).

This procedure was applied to the five different sets of data made available to us through the courtesy of their authors. The general scaling property was found to hold in all five sets. Results of the fits are displayed in Table I and Fig. 1. In this preliminary investigation, we can make at least two comments: First, the Ohmic approximation can be used to describe a large variety of turbulent configurations. Second, the parameters entering into the Ohmic solution seem to depend on the turbulent configuration and/or the pseudo-Reynolds number. They could then be used to ''classify'' different turbulent flows. A more detailed exploration of this classification is in preparation [27], studying variations of the parameters with the Reynolds number in a given configuration.

B. Computation of δu^0

To compute the complete structure functions, one needs the behavior of the nonuniversal part, namely, the ''most probable'' event δu_ℓ^0. We can use an idea of Benzi *et al.* [28] and use the so-called ''Kolmogorov $\frac{4}{5}$ law'' derived from the Navier-Stokes equation, where ϵ is the energy dissipation rate and ν is the viscosity,

$$\langle (\delta u_\ell)^3 \rangle = -\tfrac{4}{5}\epsilon \ell + 6\nu \partial_\ell \langle (\delta u_\ell)^2 \rangle, \tag{20}$$

and the relation derived from the log-normal statistics,

$$\langle |\delta u_\ell|^n \rangle = (\delta u_\ell^0)^n e^{n^2 X_1 + f_n}, \tag{21}$$

where f_n is a scale independent integration constant. Equations (20) and (21) deal, respectively, with algebraic and absolute values structure functions, which for odd n are of course different. It is not clear in general whether they have the same behavior. For instance, Stolovitzky and Sreenivasan [29] showed differences at large n. However, it seems that, for $n=3$, both structure functions behave almost identically, up to a proportional constant. This has been checked, e.g., experimentally in three-dimensional (3D) turbulence by Benzi, Ciliberto, and Chavarria [30], and numerically in 2D turbulence by Babiano, Dubrulle, and Frick [31]. To compute δu_ℓ^0 from Eq. (20), we then only need the value of the proportionality constant between $\langle (\delta u_\ell)^3 \rangle$ and $\langle |\delta u_\ell|^3 \rangle$, which can be computed experimentally.

V. CONCLUSION

In the present work, we used the idea that the behavior of the structure functions could be divided into two distinct parts: a nonuniversal part, depending on the shape of the most probable events, and a universal part, characterized by reduced structure functions. Previous papers [4,5] classified the possible variations of the scaling exponent with the *order of the moment*. This classification covers the set of infinitely divisible laws, and includes as special cases the log-normal model [9,8], the phenomenological model of She-Leveque [11], the thermodynamical model of Castaing [32], the β model [33], and the Kolmogorov solution [9].

Recent experimental results seem to indicate that turbulence is characterized by log normal [3], or very near log-normal [14], statistics. Therefore, in the present paper, devoted to the variation of the scaling exponents with the *scale*, we apply our unifying formalism only on this log-normal statistics. Our conclusions can be summarized as follows:

(i) The nonuniversal part can be set aside as a first step (while the universal part is determined), then finally computed from the Kolmogorov $\frac{4}{5}$ law.

(ii) We have tried to determine the universal contribution using only symmetry arguments, and a formalism with the same symmetry as electricity (Sec. II B and the Appendix).

(iii) In particular, the general scaling of the reduced structure functions observed in turbulence by Benzi *et al.* [15] can be explained by the conservation along the scale of a general impulsion (Secs. II D and II C). This conservation law is

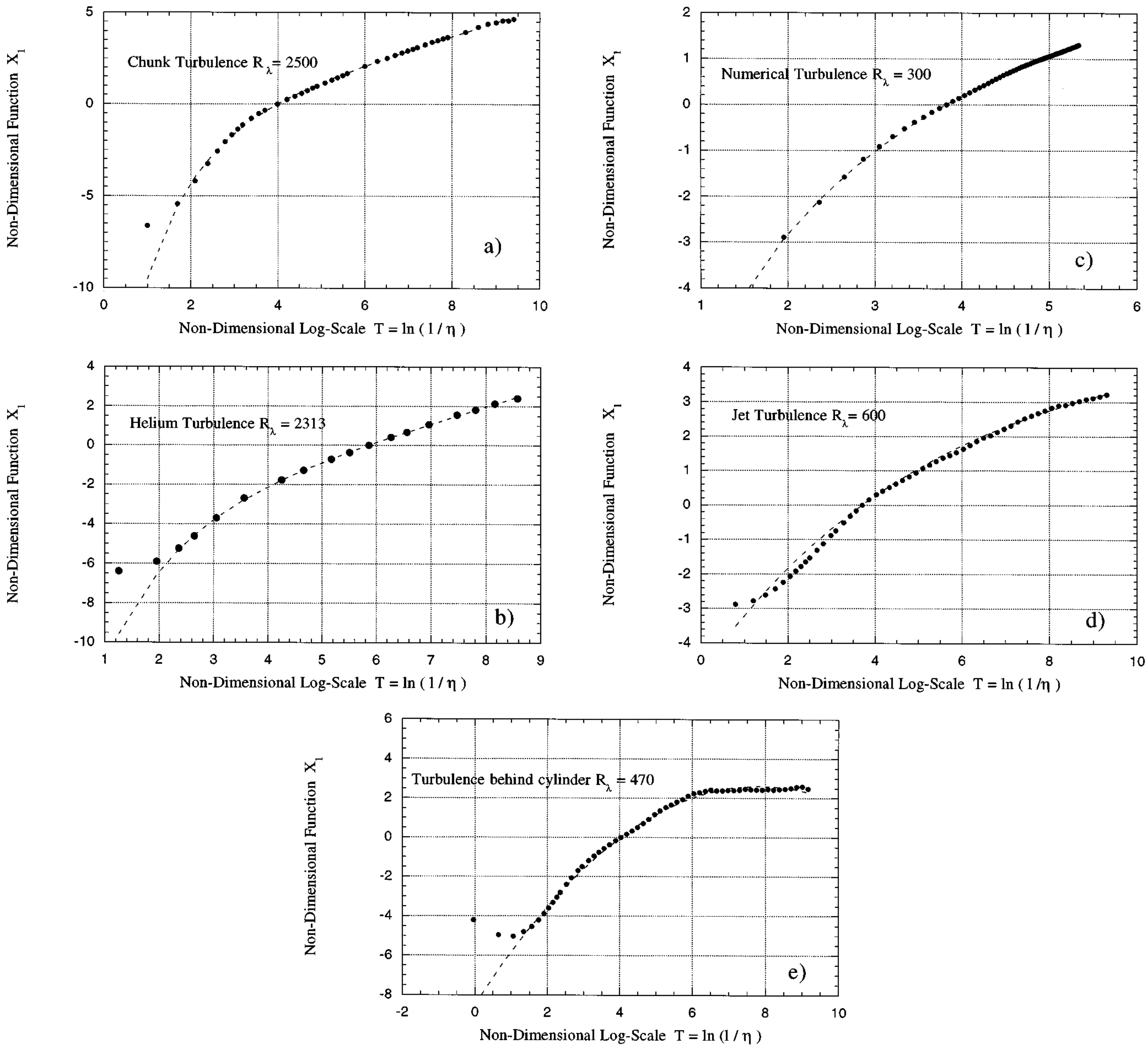

FIG. 1. Comparison between the function $X_1(T)$ computed from real data (filled circles) and a fit (dotted line) to an Ohmic solution obtained using the procedure described in the text. X_1 is the logarithm of a reduced (adimensioned) structure function, and T is a nondimensional log scale, defined as the logarithm of the scale ℓ divided by the Kolmogorov scale η. Our most general solution (15c) fits well data of turbulence in a wind tunnel (a), courtesy of Y. Gagne; in a helium tank with counter-rotating cylinders (b), courtesy of F. Belin, P. Tabeling, and H. Willaime; and in numerical simulations of Navier-Stokes equations via spectral method (c), courtesy of M. Meneguzzi. The ''stretched exponential'' solution (17) provides the best fit to data of a turbulent jet (d), courtesy of S. Ciliberto. The ''degenerate'' solution (16), i.e., with a zero ''conductivity'' σ, provides the best fit to data of turbulence behind a cylinder (e), courtesy of S. Ciliberto.

associated with the scale symmetry of amplitude fluctuations. It therefore can be seen as a generalization of the energy conservation which is the central hypothesis of classical phenomenological theories derived from the Kolmogorov 1941 theory [9].

(iv) In such case, the reduced structure functions depend only on one function of the scale. This function could in principle be computed; the pertinent shape of the equations is established in Sec. III A.

(v) Section III A and Eq. (14) explain the only possibility of establishing scale-symmetric equations through a regular perturbation development around the limit of infinite Reynolds number.

(vi) Solutions of these scale-invariant equations, but with finite boundary conditions, play the same generic role as power laws do for boundary conditions rejected at infinity. As an example, a solution is explicitly computed in the linear (or '' Ohmic'') closure (Sec. III B): we find a stretched exponential with [Eq. (15c)] or without [Eq. (18)] a power-law prefactor.

(vii) The corresponding solutions potentially describe a large variety of turbulent flows, from numerical simulation to wind tunnel turbulence, including flow past a cylinder, jet turbulence, or turbulence in helium. Whether other types of turbulence also follow the Ohmic behavior is an open question.

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APPENDIX: POSSIBLE LAGRANGIAN IN THE LOG-POISSON CASE

This appendix uses the results and notations of our companion paper to determine the possible shapes of Lagrangian formalisms compatible with scale-symmetry requirements. In the present paper, we developed only the simplest case, namely, the log-normal statistics. However, our formalism applies equally well to other statistics. In this appendix, we examine the case of log-Poisson statistics [2].

In this case the large-scale–small-scale symmetry breaking parameter $\Lambda=1$, and there is only one codimension of the most intermittent structure, which is finite and equal to $C_-=C/2$. The similarity factor is $\Gamma(\dot{X})=(1-\dot{X}/C_-)^{-1/2}$. A particle motion is determined by the Lagrangian

$$\mathcal{L}=-\frac{MC^2}{\Gamma(\dot{X})}+eA-e\Phi\left(1-\frac{\dot{X}}{C}\right). \tag{A1}$$

This is an analog of electromagnetism reduced to a bidimensional space-time (T,X), i.e., where only a scalar electric field E and no magnetic field are allowed. Here, e and M are the analog to the charge and the mass of the particle, respectively, so that the dynamics of the particle only depends on the scalar coupling constant $\alpha=e/MC$. The field E derives from the potentials A and Φ via

$$E=-\partial_X\Phi-\partial_{CT}\Phi+\partial_X A. \tag{A2}$$

Note that α, and more surprisingly E, are scalar, independent of the referential.

There are an energy and a generalized impulsion associated to this Lagrangian:

$$\mathcal{P}=\partial\mathcal{L}/\partial\dot{X}=\Gamma MC+e\Phi/C,$$
$$\mathcal{E}=\Gamma MC^2\left[1-\frac{\dot{X}}{C}\right]+e[\Phi-A]. \tag{A3}$$

Whenever the symmetry by translation along X holds, one can set $\partial_X=0$ in Eqs. (A2) and (A3). The generalized impulsion $\mathcal{P}$ is conserved along the scales

$$\partial_T\mathcal{P}=0, \tag{A4}$$

i.e., the dynamics is equivalently given by the Euler-Lagrange equation

$$MC\frac{d\Gamma(\dot{X})}{dT}=\frac{d(\mathcal{P}-e\Phi/C)}{dT}=eE, \tag{A5}$$

which is more simply written as

$$d\Gamma=\alpha E\ dT.$$

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[1] Instead of δu_ℓ, one can, indifferently, choose $\delta u_\ell/\ell$, analog to a component of a velocity gradient [2]. Wavelet analysis suggests a much better choice, yet not as widespread as it deserves, see Ref. [3].
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[16] In the formalism of the companion paper, the log-normal case is simply the non-relativistic case, obtained in the limit of infinite C_- and C_+. The composition law for exponents is Galilean, i.e., is simply an addition; the Dynamics is Newtonian-like, see below. The log-Poisson case would not be so trivial; see the Appendix.
[17] Again, think of the analogy with relativity: special relativity selects Maxwell's as the interaction which respects the symmetries; the dynamical equation $dv/dt=eE/m$ is still valid even in the nonrelativistic $v/c\rightarrow 0$ limit.
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[19] At much smaller scales, $T<|\sigma^{-1}|$, if σ is negative. In electricity, the product JE is positive; the argument is that the system has a positive temperature, and that its entropy must grow with time [18]: this constrains σ to be positive. Here the argument is less clear. In three-dimensional fully developed turbulence, the upper cutoff plays the role of the initial condition. In other types of turbulence, why not accept a negative σ, e.g., when the effective viscosity is negative and energy cascades toward large scales? More generally, $\sigma<0$ for any problem where the physically relevent length scale is the lower cutoff.
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