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Towards an universal classification of scale invariant processes

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Abstract. We consider fields which take random values over several decades. Starting from physical examples, we postulate that *scale is not an absolute quantity.* We then establish the equivalence between two existing approaches based on scale symmetry arguments as general as possible. This yields a classification of log-infinitely divisible laws, possibly universal. The physical significance of the parameters entering in the classification is discussed.

PACS. 11.30.-j Symmetry and conservation laws

1 Introduction

The properties of many physical systems are described by processes (= random fields) taking their value over a few decades; velocity increments in turbulence, energy released by earthquakes, market rates in economics are often quoted examples [1]. Such processes can exhibit different statistical properties at different length, time or energy scales ℓ.

In preceding papers, we separately tried to classify these statistics according to their dependance on scale, using only very general symmetry arguments. One of us suggested to characterize scale symmetry as a gauge invariance, *i.e.* a property of invariance through changes of unit standards [2]; two of us suggested to link scale symmetry with a matricial group structure [3,4], following an analogy with relativistic mechanics inspired by Nottale [5]. Both methods are similar in spirit but solutions exhibited a few marked discrepancies. In this letter we show where they arose from and how within the same fundamental postulates, they mutually enlarge and enrich each other.

The connection between the two different approaches might be achieved either by recasting the gauge invariance formalism in a matrix form or by rederiving the approach based on matrices labelled by a parameter. We chose the latter in the following. However, we relax the constraint that the group law for this parameter should be a simple addition. This allows for more different statistics, makes both methods consistent and opens the possibility of a complete, universal classification of processes varying over many decades.

We define a scale invariant system as a system undergoing scale invariant interactions. Mathematically speaking, it is thus governed by equations which keep the same shape at any scale. This is the operational description of "scale covariance", as discussed in [6]. This scale symmetry can be exact [1].

Now, consider a random process ϕ which is a physical solution of these "scale covariant" equations[1]. ϕ can be any scalar quantity, taking its value over several decades, such as one can define its value ϕ_ℓ measured at scale ℓ [8]. A common example is isotropic turbulence, where ϕ_ℓ can be the longitudinal increment of velocity over a distance ℓ. In what follows we assume that ϕ_ℓ can be experimentally measured and receive a statistical description, *i.e.* that it admits an underlying probability distribution function. It may be described either by its histograms or by its moments. Experimentally, both descriptions are used in order to track the probability distribution function. They are different in essence, they have different sensitivities to noise and to extreme deviations, but they carry the same mathematical and physical information. For instance, the local exponents $\zeta(n,\ell) = d\log\langle\phi_\ell^n\rangle/d\log\ell$ derived from moments characterize the scale properties of the probability distribution. Also, for any realizable value

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[1] In the sequel, we consider only positive processes, to be able to deal with its logarithm or non-integer powers. Extensions in the complex plane permit generalizations to negative processes [7].

of the random process ϕ_*, there is a real s such that $\langle f(\phi)\,\phi^s\rangle/\langle\phi^s\rangle = f(\phi_*)$ if the function f is non-zero and regular enough [10]. This connection between values and moments, often understated, will turn particularly useful when we define the variable X below.

Imagine at first an ideal, mathematical solution ϕ_ℓ which is itself "scale-invariant"; in the sense that it does not break the symmetry of the equations. Classically, this is evidenced upon rescaling of scale by a factor λ, and of process by a factor $\mu = \lambda^h$, where h is any number, leading to a process having the same statistical properties than the original one. Then, although $\mu\phi_{\lambda\ell}$ and ϕ_ℓ do not not necessarily have the same individual realizations, their distribution functions are the same up to all moments. This usual definition (see *e.g.* [11]) introduces a family of natural operators, the dilation operators:

$$S_{\lambda,\mu}: \ell \to \lambda\ell; \quad \phi_\ell \to \mu\phi_\ell, \tag{1}$$

which simply compose as: $S_{\lambda,\mu} \circ S_{\lambda',\mu'} = S_{\lambda\lambda',\mu\mu'}$. Processes ϕ_ℓ statistically invariant under such dilations have nth moments which go as power laws $\ell^{\zeta(n)}$ of the scale, *i.e.* they check $\partial\zeta/\partial\ell = 0$. This corresponds to the multifractal description [11,12] according to which the exponents $\zeta(n)$ are related to the set of exponents h by the Legendre transformation $\zeta(n) = \min_h(nh + D(h))$, where $D(h)$ is related to the probability of occurence of the exponent h. Then $h = d\zeta(n)/dn$.

However, this classical analysis suffers from a severe drawback. As usual, symmetric equations with symmetry-breaking boundary conditions admit symmetry-breaking solutions. Scale independent objects, in the sense we have defined, exist only in an idealized, infinite-size mathematical limit. Scale invariant equations are widespread but *any* real system has a lower and an upper cut-off which break the dilation symmetry. Moments are thus never exact power laws: ζ always varies with ℓ, at least near cut-offs. According to common experience, even when a $\log\langle\phi_\ell\rangle$ *versus* $\log\ell$ diagram looks straight, a fit with a straight line is not necessarily close to the idealistic properties of a scale-invariant system [9] with no cut-off. This physical constraint cannot be treated as perturbations for pure scale dilation operators and requires to find more general operators. However, as the scale symmetry of the equations themselves need not be broken by cut-offs, we may still apply our basic postulate [2–4,6]: *the scale is not an absolute quantity* and can depend, according to one's point of view, on the observer or on the realization of the process.

This is reminiscent of Nottale's approach [5]. Like him, we want to formalize this freedom by exhibiting a group structure for scale transformations [3]. This we do by defining successively (i) a variable T going as the log of the scale; (ii) a variable X going as the log of the process; (iii) a coupling between them, stating that the scale is not an absolute quantity but can depend on the process. This will be physically discussed in Section 5.2.

2 Formalism

We first define a log-scale coordinate characterizing the measurement of a scale with respect to any scale unit ℓ_0, *via*:

$$T := \frac{\ln(\ell/\ell_0)}{\ln(K)} = \log_K(\ell/\ell_0),$$

where K is the basis of the logarithm and ℓ_0 is an arbitrary unit. Scale symmetry then implies homogeneity in this log-coordinate.

In a similar way, we introduce a log-coordinate characterizing the measurement of a given realization of the random process ϕ_ℓ, at this scale, with respect to any "field unit" ϕ_0 (possibly random and/or scale dependent [3]): $X := \langle\, f(\phi_\ell/\phi_0)\ (\phi_\ell/\phi_0)^s\,\rangle/\langle(\phi_\ell/\phi_0)^s\rangle$, where s is a real number. We impose that f must be a linear function of the logarithm, so that a dilation on ϕ appears as a translation on X. We thus write:

$$X = \left\langle \log_Q(\phi_\ell/\phi_0)\right\rangle_{\mathcal{R}_s},$$

where again the basis Q of the logarithm and the scale-dependent unit ϕ_0 are chosen by the observer. The notation $\langle\ \rangle_{\mathcal{R}_s}$ is used to recall that the average is taken with respect to the weighted probability distribution $\phi^s/\langle\phi^s\rangle$, which plays the role of a "reference frame" (see *e.g.* [4,13,14]). In these notations, $S_{\lambda,\mu}$ appears as a translation of vector $(\ln\lambda, \ln\mu)$ applied on the origin of coordinates (T, X).

3 The scale transformation

We now want to relate two measurements T, X, performed at a given scale, in a given realization, to similar measurements performed at a different scale, in another realization T', X', that is varying ℓ to ℓ' and s to s'. We stress here that there are three equivalent points of view: the same numbers X' and T' can be obtained by independent variations of either ℓ and s, as adopted here; or ℓ_0 and ϕ_0, as adopted in [2–4]; or even by varying K and Q, as discussed in [4].

Formally, as shown in preceding papers [3,4], symmetry requires that both sets of log-coordinates (T, X) and (T', X') are coupled through a linear transformation of determinant one. For the exponents to be "relative", physical relations must be invariant under a change of logarithm basis. Physical laws can therefore link the values of different exponents but never single out a particular value. This imposes to consider a group structure for the transformations whose shape [3] is selected in analogy with special relativity [4,5]:

$$\begin{pmatrix} T' \\ X' \end{pmatrix} = \overset{=}{a}_{ij} \begin{pmatrix} T \\ X \end{pmatrix} = \Gamma(V) \begin{pmatrix} 1 - V(1/C_+ + 1/C_-) & V/C_+C_- \\ -V & 1 \end{pmatrix} \begin{pmatrix} T \\ X \end{pmatrix},$$

$$\Gamma(V)^{-2} := 1 - V(1/C_+ + 1/C_-) + V^2/C_+C_-. \tag{2}$$

• There appears a parameter $V = V_{\mathcal{R}_s/\mathcal{R}'_s}$, characterizing the relative scaling exponent, *i.e.* the scaling of the first coordinate system with respect to the second. Applying (2) successively from ϕ_ℓ to $\phi'_{\ell'}$, then from $\phi'_{\ell'}$ to $\phi''_{\ell''}$, one obtains a group law for V:

$$V \otimes V' := \frac{V + V' - VV'(1/C_+ + 1/C_-)}{1 - VV'/C_+C_-} \,. \qquad (3)$$

The law $\otimes$ is commutative and exhibits two fixed points $C_\pm$. We stress that it is the only possible group law isomorph to the addition of real numbers in the interval bounded by $C_\pm$ (see [3]).

• The two fixed points $C_\pm$ depend only on the random process itself; for instance, $1/C_+ + 1/C_- \neq 0$ iff the symmetry between small and large scales is broken. They play an essential role since they classify the possible statistics [3].

• The novel scale symmetry operators appear naturally as the analog of Lorentz boosts:

$$S_{\lambda,\mu,V}\colon\ \ell \to (\lambda\ell)^{a_{11}(V)}(\mu\phi_\ell)^{a_{12}(V)}, \quad \phi_\ell \to (\lambda\ell)^{a_{21}(V)}(\mu\phi_\ell)^{a_{22}(V)}. \qquad (4)$$

This family includes, but does not reduce to, the former operator $S_{\lambda,\mu} = S_{\lambda,\mu,V=0}$ acting as dilation on scales (or translation on log-scales). These novel operators have no special reasons to select processes with scale invariant moments ($d\zeta(n)/d\ell = 0\ \ \forall n, \ell$): see [4] for a discussion of the variation of ζ with the scale ℓ.

4 Possible statistics

In what follows we rather want to determine how these novel operators select possible statistics considered at a given scale ℓ. This means that we want to relate the values of the moments $\langle \phi_\ell^n \rangle$, or equivalently of the $\zeta(n)$, for different values of n. We thus fix ℓ, and when derivation is needed (for instance to define ζ knowing ϕ_ℓ) we consider only an infinitesimal neighbourhood of it. In this paper we take ζ and V as functions of the sole variable n, as in [2,3,15].

4.1 Correspondence between histogram and moments

We then must precise the link between X and V. This depends of course on the specific choice of ϕ_0.

For instance, if ϕ_0 is chosen to be a number, equal to the log-average of ϕ_ℓ (*i.e.* $\ln\phi_0 = \langle \ln\phi_\ell \rangle$), then:

$$V := V_{\mathcal{R}_n/\mathcal{R}_0} = \partial_n \zeta - \partial_n \zeta|_{n=0}. \qquad (5)$$

In that case $V(0) = 0$ and $V = h - h_0$, so that the link with the multifractal description is easy. This way of rendering ϕ_ℓ dimensionless is a judicious choice [4] and we examine it in the next subsection.

More generally, the correspondence between histogram and moments appears here as a bijection $n \to V(n)$ defined on the interval $[n_-, n_+] = V^{-1}[C_-, C_+]$. It unambiguously defines an internal composition law $\tilde{\otimes}$ on $[n_-, n_+]$, through a transport of the group structure $\otimes$:

$$V(n \tilde{\otimes} p) := V(n) \otimes V(p), \quad n \tilde{\otimes} p := V^{-1}(V(n) \otimes V(p)). \qquad (6)$$

Physically, this means that the probability distribution admits convergent moments of order n for all values of n in that interval, and divergent moments for other values of n.

The correspondence established through equation (6) is the main result of this paper, as will now become clearer: in fact, it enables to classify the statistics of scale invariant random process, *i.e.* of the possible shapes for $\zeta(n)$, on universal grounds.

Classification proceeds as follows. Consider equation (6) with $n = p = 1$. This provides a recursive equation which allows the computation of V for any integer m, and then, by continuation, on any real number, namely:

$$E(m) = 1 \tilde{\otimes} 1 \tilde{\otimes} ... \tilde{\otimes} 1 = 1^{\tilde{[m]}}, \quad V(E(m)) = V(1) \otimes V(1) \otimes ... \otimes V(1) = V(1)^{[m]}. \qquad (7)$$

Here the notation $[m]$ (resp. $\tilde{[m]}$), stands for the mth iterate *via* $\otimes$ (resp. $\tilde{\otimes}$): we have introduced the notation E to generalize the exponentiation to m real and not integer, defined using infinitesimal n and p.

Inverting (5), we get:

$$\frac{d\zeta(n)}{dn} = h_0 + V(1)^{[E^{-1}(n)]}. \qquad (8)$$

This provides a symbolic representation of the possible shape for the scaling exponents, as a function of the composition laws, *i.e.* in particular of the four fixed points $n_\pm, C_\pm$. The dependence on ℓ, here understated, can be integrated within these parameters according to the observer's choice, as discussed in Section 5.2.

4.2 A good method to adimension the field

Let us focus on the case where ϕ_0 is chosen to be the log-average of ϕ_ℓ. A possible shape for $\tilde{\otimes}$ might be given by (3) with the parameters $n_\pm$ instead of $C_\pm$:

$$n \tilde{\otimes} n' = \frac{n + n' - nn'(1/n_+ + 1/n_-)}{1 - nn'/n_+n_-} \,. \qquad (9)$$

More generally, as the composition law on the interval $[n_-, n_+]$ isomorph to the addition of real numbers is unique [3], we know that the shape of $\tilde{\otimes}$ can be reduced to (9) by some non-linear application involving $n_\pm$ as fixed points. We note that this application is determined by the isomorphism $n \to V(n)$.

Let us focus here on the case where (9) is satisfied. It is then technically possible to compute the values of the $\zeta(n)$, parametrized *e.g.* by the value of $\zeta(1)$: the set of equations (3, 6, 7, 9) is a mathematically well posed problem. The possible shapes of the functions $E(n)$ and $V(n)$ as a function of $C_\pm$ or $n_\pm$ are given in [3]. We list below but a few examples:

- log-Poisson: this case [13,16,17] was already obtained in [2,3]. It corresponds to C_- finite, $C_+ = \infty$, $n_- = -\infty$, $n_+ = +\infty$. It reads:

$$\zeta(n) = n\,(h_0 + C_-) \;+\; \frac{C_-}{\ln\beta}\,(1-\beta^n);$$

  the parameters are here h_0, C_- and $\beta = 1 - V(1)/C_-$, or equivalently $\zeta(1)$;
- self-similar: this limiting case of the previous one is obtained for $C_- = 0$, and reads:

$$\zeta(n) = n\,h_0;$$

- log-normal: this is again a limiting case of the log-Poisson, with $C_- \to \infty$; it corresponds to:

$$\zeta(n) = n\,h_0 + n^2\ \left(\zeta(1) - h_0\right);$$

- a log-Levy like distribution: this case, also sometimes called "truncated log-Levy" [18], was obtained in [2]. It corresponds to C_- finite, C_+ infinite, n_+ finite and n_- infinite (or *vice versa*). It is defined only for $n < n_+$ and yields:

$$\zeta(n) = n\,(h_0 + C_-) - C_-\frac{n_+}{\alpha}\left[1 - \left(1 - \frac{n}{n_+}\right)^{\alpha}\right].$$

  Here, α along with C_-, n_+, h_0, make four parameters, one of which could be replaced by $\zeta(1)$.

We have chosen these simple examples because they are precisely those obtained in [2] using a gauge invariance method. This shows that the matrix approach presented in [3] can be made consistent with the gauge approach, provided the constraint of additive structure in n is relaxed.

4.3 Coupling between field and scales

The above examples correspond to a case where T' is decoupled from X, *i.e.* where there is no effect of field on scale, as requested in [2]. This point is further discussed in Section 5.2. Formally, this is equivalent to the condition that the off-diagonal matrix element $a_{12} = -V\Gamma/C_+C_-$ of (2) vanishes whatever V, *i.e.* $C_+C_- \to \pm\infty$.

If this constraint is removed, additional solutions may be obtained [3,15]. For instance, consider the $T \to -T$ (large $\to$ small scale) symmetry preserving case where C_- and $-C_+$ are equal and finite: with $n_- = -\infty$, $n_+ = +\infty$, one obtains

$$\zeta(n) = n\,h_0 + \frac{C_-}{\gamma}\,\ln(\mathrm{ch}(n\delta)),$$

where $\delta = \ln\sqrt{1 + 2/\beta} \neq 0$, h_0 and C_- are three parameters.

We thus clarified the connection and differences between the methods used in [2,3]. This was one of the main goals of the present paper. We can now proceed further and explore potential implications and applications of the unified method proposed here.

5 Applications and implications

5.1 Link with the log-infinitely divisible laws

All above examples belong to the family of the so-called "log-infinitely divisible laws". One question which is natural to ask now is whether the representation (8) in fact just characterizes the complete set of log-infinitely divisible laws. We have no rigorous answer to that, but a hint that it might indeed be the case. The Poisson law is the brick from which one can build all infinitely divisible laws [19]. Let us then consider what happens when C_+, say, is infinite. From the results obtained in [3], we may then write (8) as:

$$\zeta(n) = n\,h_0 + C_-\int_0^n \left(1 - \beta^{E^{-1}(p)}\right) dp, \qquad (10)$$

where as in the preceding section we note $\beta = 1 - V(1)/C_-$. Changing variable from p to $q = E^{-1}(p)\ln\beta/n$, we get:

$$\zeta(n) = n\,h_0 + \int_0^{E^{-1}(n)\ln\beta/n} \left(1 - e^{nq}\right)\, dF(q), \qquad (11)$$

where $dF(q) \equiv C_- dE(nq/\ln\beta)$. Upon taking the limit $\beta \to \infty$ (*i.e.* $C_- \to 0$), one gets a representation of $\zeta(n)$ mimicking the Levy-Khinchine general representation of infinitely divisible laws [19].

We thus conjecture that the representation (8) is in essence equivalent to the Levy-Kinchine representation, *i.e.* that we obtain by our method all log-infinitely divisible laws. Is this really a coincidence? Of course, not. The starting point of the obtention of the composition law (3) is, according to [3], the scale symmetry invariance, which translates into an assumption of homogeneity in variables X, T. The starting point of the classification of log-infinitely divisible laws is precisely that the variable X has stationary increments, *i.e.* is homogeneous. That the statistics of any scale invariant process is an infinitely divisible law is already known [20] and might date back to Kolmogorov [21]. The novelty resides in fact in two points.

First, it is shown here that the only physical ingredient leading to the law is the homogeneity in variable log, *i.e.* the scale symmetry of interactions between fluctuations of different amplitudes. This demonstration does not require to invoke any notion such as continuous cascades, or multiplicative process. Such remark has an important consequence: it means that this symmetry can rule physical systems in which coupling between scales (*e.g.* energy transfer) is not local in the scale space.

Second, we obtain an universal classification of the log-infinitely divisible statistics. In fact, there are two composition rules (Eqs. (3, 6)) and thus four relevant fixed points. Two of them are minimum and maximum scaling exponents, linked to the codimension of the rarest events [3]: they are thus topological quantities. The two remaining fixed points are the minimum and maximum orders of the convergent moment: they are linked to the moment generating function. We classify the various possible statistics according to the number of the fixed points: whether they are finite or infinite, and whether they are distinct or equal [2,3]. The residual degree of freedom resides in the choice of the isomorphism $V(n)$.

5.2 Selection of the parameters of the classification

The previous classification depends on four parameters $n_\pm$ and $C_\pm$, and one isomorphism $V(n)$. It is of practical interest to determine how these parameters depend on the system, and which parameters will be selected from a given experimental or numerical signal.

• The first influence emphasized by our derivation comes from the experimentator who records the signal, the "observer" [2,3]. What our covariance derivation really stresses is that a same signal can lead to different statistics, depending on the way adopted to record it. That leads back to a trivial remark: if two persons are observing the same physical signal, and if the first one chooses to record the data corresponding to a physical quantity ϕ, then the second one is free to record and analyze the signal ϕ^2, $\sqrt{\phi}$ or e^ϕ. Of course, they will not obtain the same statistics[2] (if ϕ is Gaussian, certainly e^ϕ is not Gaussian). But both statistics are related *via* a simple isomorphism, respectively $n \to 2n$, $n \to n/2$ and $n \to n^2$. This trivial example, which we will use again below, is an illustration that a scale invariant statistics can only be classified up to an isomorphism (we called it V), which depends either on the choice of the field itself, or on the "units" ϕ_0 and ℓ_0 chosen to measure it.

• Consider now the parameters $C_\pm$. For a given choice of the "units" ϕ_0 and ℓ_0, they characterize the coupling between the field ϕ/ϕ_0 and the scale ℓ/ℓ_0. Indeed, we note that when C_+C_- (*i.e.*, say, C_+) is infinite, the field is coupled to the scale, but the scale itself is not coupled to the field (X' depends on T, but T' is independent of X). So, the knowledge *a priori* of the coupling between field and scale can help to constrain the possible statistics in a given system. In most physical experiments, one considers that the field depends on the scale, but that the scale usually does not depend on the field. For simple scale independent units, this *a priori* selects only four possible statistics: the self-similar, the log-Poisson, the log-normal or the truncated log-Levy.

One should however be cautious in drawing such conclusion because the feedback of a field to the scale of measurement can be lurking at the stage of the physical measurement, or later when analyzing data. For instance, consider an experiment of hydro-dynamical turbulence. A hot-wire measurement of velocity fluctuations at a certain scale itself depends on how the velocity fluctuates at other scales. The signal measured at one scale will be more or less sensitive to the global structure of the velocity field according to the number, disposition and vibrations of each hot-wire probe. Two observers can thus acquire different statistical signal from the same turbulent bath depending on how their probe reacts to the field.

Also, they can introduce the feedback, not during measurement, but later, during the analysis. For instance, a temporal signal can be translated into a measurement of fluctuations at different space scales if one assumes, following Taylor, that the fluctuations at small scales are frozen and carried away by large scale currents; this amounts to writing at each scale ϕ_ℓ as $\langle\phi_\ell\rangle + \delta\phi_\ell$ where fluctuations $\delta\phi_\ell$ are assumed to be much smaller than the average $\langle\phi_\ell\rangle$. It would therefore not be surprising to observe a different type of statistics in *e.g.* jet turbulence (typically 10-20 percent of rms turbulence) where Taylor approximation barely affects the signal analysis, and in another turbulence where Taylor approximation is strongly violated and thus introduces an unavoidable feedback of field on scale. Such effect might already have been observed by Pinton and Labbé [22].

• We finally turn to the last parameters $n_\pm$, *i.e.* the range of non-divergent moment orders[3]. In thermodynamical analogies of multifractals, they correspond to critical temperature(s), which need not be unique. This analogy suggests that these parameters could be characteristic of the system. Especially, there is strong evidence that they depend on physical cut-offs and that the only possible statistics in an infinite size system is the self-similar statistics [23]. Dubrulle and Andreotti [24] have studied the generic scale symmetry breaking of a self-similar statistics; they find a class of solutions including the truncated log-Levy-laws, thus proving that finite size effects influence the values of $n_\pm$. Other external parameters in the system are also probably influent: Arneodo *et al.* [25] see a tendency for statistics in turbulence to approach the log-normal statistics as the Reynolds number is increased. These kinds of analysis certainly deserve more attention.

6 Conclusion

To summarize, we generalized our preceding studies of fields which take random values over several decades.

[2] Note also that the choice of system of units and sub-units imbeds exactly the same degree of arbitrariness: measuring the same ϕ, using a different variation of the unit $\phi_0(\ell)$ with the scale, also leads to different statistics for ϕ/ϕ_0. This effect has luckily disappeared due to the widespread use of interchangeable international systems of sub-units growing in geometrical progression of ratio $Q = 10$ [4].

[3] Note that in this paper we do not consider accidental divergences of moments due to the prefactor of the scale dependence. That these prefactors do or do not diverge of course depends on the choices made during the recording and the analysis of the signal.

Starting from the postulate that scale is not an absolute quantity, we reach a classification of log-infinitely divisible laws with symmetry arguments as general as possible. This classification at least covers previously published statistics and is probably universal. We discussed the physical significance of the parameters entering the classification, and their dependence on the characteristics of the system. The open problem is now to compute the values of the parameters in a given system.

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